

Infinite Bottles of Beer
Mathematical Concepts with Epsilon Pain
Or: A cantor approach to Cantorian arithmetic and other
mathematical melodies

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with Considerable Help from His Friends

revised late July 2011

(Started in 1987; revised 1990-2011. Copyright (C) 1987-2011, Donald Byrd. Note: an earlier and much shorter version of this article appeared in the September 2010 issue of *Math Horizons*, pp. 16-17.)

A classic summer-camp song – at least that’s the way I think of it – goes like this:

A hundred bottles of beer on the wall, a hundred bottles of beer;
If one of those bottles should happen to fall,
99 bottles of beer on the wall.

99 bottles of beer on the wall, 99 bottles of beer;
If one of those bottles should happen to fall,
98 bottles of beer on the wall.

98 bottles of beer on the wall, 98 bottles of beer;
If one of those bottles should happen to fall...

...

One bottle of beer on the wall, one bottle of beer;
If that bottle should happen to fall,
no bottles of beer on the wall.
(The End)

The idea, of course, is to entertain kids on a long trip with a long song. For a longer trip, make a longer song by starting with 200 or 300 bottles. Some years ago, the following playful variation – for *really* long trips! – occurred to me:

Infinite bottles of beer on the wall, infinite bottles of beer;
If one of those bottles should happen to fall,
infinite bottles of beer on the wall.

Infinite bottles of beer on the wall, infinite bottles of beer;
If one of those bottles should happen to fall,
infinite bottles of beer on the wall.
(etc.)

But then I thought, why stop with countable infinities? So another version occurred to me, and another of a somewhat different kind. I started writing them down, adding contributions from friends I occasionally received. Somewhere along the way, it occurred to me that these “songs” actually carry some pedagogical value in that they make relatively painless demonstrations of mathematical concepts that people often find hard to understand. The full collection appears below; all are by me unless stated otherwise (and “DAB” = me). Note that most of them, namely those that say “(etc.)” at the bottom, are non-terminating; in one version, the very first line is non-terminating. While the vast majority are mathematically inspired, a few are based on ideas of other types. I have no doubt many more entertaining and illuminating variations are possible. Perhaps you’d like to contribute one or more?

A last (and particularly light-hearted – fair warning) comment. Some might consider these “bottles of beer” songs to be unsuitable for students below drinking age. Perhaps so, but Larry Lesser (personal communication, September 2010) quotes “mathemagician” Art Benjamin as suggesting that this can be handled by putting a radical sign over the word “beer”, thus turning it into “root beer”!

Sequences or Functions of Time and Visualization

Of course, these variations describe sequences or, in some cases, functions of time. The “standard” version has its range in \mathbf{N} , its value monotonically decreasing from a finite positive value to zero in a finite number of steps. Some of the below variations behave the same way, and just describe it in unusual ways – for example, the differential version and the Euler’s identity basic version. But many of these versions behave very differently. Some start at infinity in \mathbf{N} or \mathbf{R} ; some have rational, real or complex values that approach zero only asymptotically, if at all; some oscillate; etc.

It’s illuminating to imagine graphing some of the versions on the real line or the complex plane, and students might enjoy doing so. However, it’s not easy to see how to graph quite a few of them, e.g., the continuum-hypothesis version, or many of those involving infinite values!

Historical Note

I’m certainly not the only person to have thought of extending the song to infinity. As of 4 May 2010, Google showed a few thousand matches total for “infinite bottles of beer on the wall” and “aleph-null bottles of beer on the wall”, and a few dozen for “aleph-one bottles of beer on the wall”. In fact, an aleph-null version has actually been published before (Renteln and Dundes, 2005).

Another way to extend the song to infinite length is simply to not stop at zero. This has been described as a version intended to annoy listeners, with the title “Negative Bottles of Beer on the Wall” (Wikipedia, 2011).

It should also be pointed out that the conventional version of the song is very often started from 99 instead of 100.

The “Bottles of Beer” Variations Collection

“Annoying” version (“Negative Bottles of Beer on the Wall”):

A hundred bottles of beer on the wall, a hundred bottles of beer;
If one of those bottles should happen to fall,
99 bottles of beer on the wall.

99 bottles of beer on the wall, 99 bottles of beer;
If one of those bottles should happen to fall,
98 bottles of beer on the wall.

98 bottles of beer on the wall, 98 bottles of beer;
If one of those bottles should happen to fall...

...

One bottle of beer on the wall, one bottle of beer;
If that bottle should happen to fall,
zero bottles of beer on the wall.

Zero bottles of beer on the wall, zero bottles of beer;
If one of those bottles should happen to fall,
Minus one bottles of beer on the wall.
(etc.)

Basic transfinite version 1:

Infinite bottles of beer on the wall, infinite bottles of beer;
If one of those bottles should happen to fall,
infinite bottles of beer on the wall.

Infinite bottles of beer on the wall, infinite bottles of beer;
If one of those bottles should happen to fall,
infinite bottles of beer on the wall.
(etc.)

Basic transfinite version 2 (a generalization of version 1):

Infinite bottles of beer on the wall, infinite bottles of beer;
If finite bottles should happen to fall,
infinite bottles of beer on the wall.

Infinite bottles of beer on the wall, infinite bottles of beer;
If finite bottles should happen to fall,
infinite bottles of beer on the wall.
(etc.)

Larger-infinity version:

Uncountable bottles of beer on the wall, uncountable bottles of beer;
If countable bottles should happen to fall,
uncountable bottles of beer on the wall.

Uncountable bottles of beer on the wall, uncountable bottles of beer;
If countable bottles should happen to fall,
uncountable bottles of beer on the wall.
(etc.)

General transfinite version:

Aleph- n bottles of beer on the wall, aleph- n bottles of beer;
If, where $m < n$, aleph- m bottles should happen to fall,
aleph- n bottles of beer on the wall.

Aleph- n bottles of beer on the wall, aleph- n bottles of beer;
If, where $m < n$, aleph- m bottles should happen to fall,
aleph- n bottles of beer on the wall.
(etc.)

Transfinite and indeterminate version (by Richard Byrd):

Infinite bottles of beer on the wall,
Infinite bottles of beer;
If infinite bottles should happen to fall,
indeterminate bottles of beer on the wall.
(The End)

Pure indeterminate version:

Indeterminate bottles of beer on the wall, indeterminate bottles of beer;
If any number, regardless of how described, should happen to fall,
indeterminate bottles of beer on the wall.

Indeterminate bottles of beer on the wall, indeterminate bottles of beer;
If any number, regardless of how described, should happen to fall,
indeterminate bottles of beer on the wall.
(etc.)

Geometric-progression version (starting arbitrarily with 2^{26}):

67,108,864 bottles of beer on the wall, 67,108,864 bottles of beer;
If half of those bottles should happen to fall,
33,554,432 bottles of beer on the wall.

33,554,432 bottles of beer on the wall, 33,554,432 bottles of beer;
If half of those bottles should happen to fall,
16,777,216 bottles of beer on the wall.
(etc.)

*** A version closely related to (indeed, contained in) the previous one:
One bottle of beer on the wall, one bottle of beer;

If half of a bottle should happen to fall,
Half of a bottle of beer on the wall.

Half of a bottle of beer on the wall, half of a bottle of beer;
If 1/4 of a bottle should happen to fall,
1/4 of a bottle of beer on the wall.
(etc.)

Generalized geometric-progression version:

m^n bottles of beer on the wall, m^n bottles of beer;
If $m^n - m^{n-1}$ bottles should happen to fall,
 m^{n-1} bottles of beer on the wall.

m^{n-1} bottles of beer on the wall, m^{n-1} bottles of beer;
If $m^{n-1} - m^{n-2}$ bottles should happen to fall,
 m^{n-2} bottles of beer on the wall.
(etc.)

Euler's identity extreme version:

$100(e^{2i\pi})$ bottles of beer on the wall, $100(e^{2i\pi})$ bottles of beer;
If $e^{2i\pi}$ bottles should happen to fall,
 $99(e^{2i\pi})$ bottles of beer on the wall.

$99(e^{2i\pi})$ bottles of beer on the wall, $99(e^{2i\pi})$ bottles of beer;
If $e^{2i\pi}$ bottles should happen to fall,
 $98(e^{2i\pi})$ bottles of beer on the wall.
(etc.)

Euler's identity basic version (by Richard Byrd):

100 bottles of beer on the wall, 100 bottles of beer;
If $e^{2i\pi}$ bottles should happen to fall,
99 bottles of beer on the wall.

99 bottles of beer on the wall, 99 bottles of beer;
If $e^{2i\pi}$ bottles should happen to fall,
98 bottles of beer on the wall.
(etc.)

Differential version (by Matthew Dovey):

100 bottles of beer on the wall, 100 bottles of beer;
If $dbeer/dt = -1$,
99 bottles of beer on the wall.

99 bottles of beer on the wall, 99 bottles of beer;
If $dbeer/dt = -1$,
98 bottles of beer on the wall.
(etc.)

Infinitesimal version:

100 bottles of beer on the wall, 100 bottles of beer;
 If ε bottles should happen to fall,
 100 bottles of beer on the wall.

100 bottles of beer on the wall, 100 bottles of beer;
 If ε bottles should happen to fall,
 100 bottles of beer on the wall.
 (etc.)

Algebraic-notation version:

n bottles of beer on the wall, n bottles of beer;
 If m of those bottles should happen to fall,
 $n-m$ bottles of beer on the wall.

$n-m$ bottles of beer on the wall, $n-m$ bottles of beer;
 If q of those bottles should happen to fall,
 $n-m-q$ bottles of beer on the wall.

$n-m-q$ bottles of beer on the wall, $n-m-q$ bottles of beer;
 If t of those bottles should happen to fall,
 $n-m-q+t$ bottles of beer on the wall.
 (etc.)

Irrational-number version:

3.1415926535 8979323846 2643383279 5028841971 6939937510...
 If one of those bottles should happen to fall,
 2.1415926535 8979323846 2643383279 5028841971 6939937510...

2.1415926535 8979323846 2643383279 5028841971 6939937510...
 If one of those bottles should happen to fall,
 1.1415926535 8979323846 2643383279 5028841971 6939937510...
 (etc.)

Increasing version (by Scott Kim):

100 bottles of beer on the wall, 100 bottles of beer;
 If one of those bottles should happen to clone,
 101 bottles of beer on the wall.

101 bottles of beer on the wall, 101 bottles of beer;
 If one of those bottles should happen to clone,
 102 bottles of beer on the wall.
 (etc.)

Explicit oscillating version:

100 bottles of beer on the wall, 100 bottles of beer;
 If one of those bottles should happen to fall,
 99 bottles of beer on the wall.

99 bottles of beer on the wall, 99 bottles of beer;
If one of those bottles should happen to clone,
100 bottles of beer on the wall.

100 bottles of beer on the wall, 100 bottles of beer;
If one of those bottles should happen to fall,
99 bottles of beer on the wall.

99 bottles of beer on the wall, 99 bottles of beer;
If one of those bottles should happen to clone,
100 bottles of beer on the wall.
(etc.)

Implicit oscillating version:

100 bottles of beer on the wall, 100 bottles of beer;
If +1 of those bottles should happen to fall,
99 bottles of beer on the wall.

99 bottles of beer on the wall, 99 bottles of beer;
If -1 of those bottles should happen to fall,
100 bottles of beer on the wall.

...

If $(-1)^n$ of those bottles should happen to fall,
 $99 + (1/2 - (-1)^n / 2)$ bottles of beer on the wall.
(etc.)

Branching version:

100 bottles of beer on the wall, 100 bottles of beer;
If one of those bottles should fall or clone,
99 or 101 bottles of beer on the wall.

99 or 101 bottles of beer on the wall, 99 or 101 bottles of beer;
If one of those bottles should fall or clone,
98 or 100 or 102 bottles of beer on the wall.
(etc.)

Summation-notation version:

Most of the others are special cases of this version.

n bottles of beer on the wall, n bottles of beer;
If m_1 of those bottles should happen to fall,
 $n - \sum_{k=1}^1 m_k$ bottles of beer on the wall.

$n - \sum_{k=1}^1 m_k$ bottles of beer on the wall,

$n - \sum_{k=1}^1 m_k$ bottles of beer;

If m_2 of those bottles should happen to fall,
 $n - \sum_{k=1}^2 m_k$ bottles of beer on the wall.
(etc.)

Arbitrary-dimensionality version:

10^n bottles of beer in an n -dimensional arrangement, 10^n bottles of beer;
If one of those bottles should happen to fall,
 $(10^n)-1$ bottles of beer in an n -dimensional arrangement.

$(10^n)-1$ bottles of beer in an n -dimensional arrangement, $(10^n)-1$ bottles of beer;
If one of those bottles should happen to fall,
 $(10^n)-2$ bottles of beer in an n -dimensional arrangement.
(etc.)

Identity version 1 (author unknown):

100 bottles of beer on the wall, 100 bottles of beer;
Take one down, put it back up,
100 bottles of beer on the wall.

100 bottles of beer on the wall, 100 bottles of beer;
Take one down, put it back up,
100 bottles of beer on the wall.
(etc.)

Identity version 2:

A hundred bottles of beer on the wall, a hundred bottles of beer;
If none of those bottles should happen to fall,
A hundred bottles of beer on the wall.

A hundred bottles of beer on the wall, a hundred bottles of beer;
If none of those bottles should happen to fall,
A hundred bottles of beer on the wall.
(etc.)

General identity version:

n bottles of beer on the wall, n bottles of beer;
If none of those bottles should happen to fall,
 n bottles of beer on the wall.

n bottles of beer on the wall, n bottles of beer;
If none of those bottles should happen to fall,
 n bottles of beer on the wall.
(etc.)

Roundoff error version (2009-10):

100.0 bottles of beer on the wall, 100.0 bottles of beer;

If .99999 of those bottles should happen to fall,
99.00001 bottles of beer on the wall.

99.00001 bottles of beer on the wall, 99.00001 bottles of beer on the wall;
If .99999 of those bottles should happen to fall,
98.00002 bottles of beer on the wall.
(etc.)

Base 2 version (modified by DAB from a version by Scott Kim, October 1997):

100 bottles of beer on the wall,
100 bottles of beer;
If one of those bottles should happen to fall,
11 bottles of beer on the wall.
(etc.)

Other mathematical variations (by Scott Kim, October 1997):

Zero bottles of beer on the wall
Zero bottles of beer;
If one of those bottles should happen to fall,
underflow.
(The End)

A hundred bottles of beer on the wall
A hundred bottles of beer;
If minus one of those bottles should happen to fall,
A hundred and one bottles of beer on the wall.
(etc.)

A hundred bottles of beer in a square
A hundred bottles of beer;
If one of the dimensions should happen to fall,
Ten bottles of beer in a line.
(etc.)

A hundred variations of a hundred bottles of beer on the wall,
A hundred variations of a hundred bottles of beer,
If one of the variations should happen to fall,
Then this variation might cease to exist.
(The End)

Fractal dimension (Cantor set) version (by Doug McKenna, Oct. 1997, revised by DAB):

n bottles of beer on the wall, n bottles of beer;
If the middle third of those bottles should fall,
 $(1/3 + 1/3) n$ bottles of beer on the wall.

$(1/3 + 1/3) n$ bottles of beer on the wall, $(1/3 + 1/3) n$ bottles of beer;
If each middle third of those bottles should fall,
 $(1/9 + 1/9 + 1/9 + 1/9) n$ bottles of beer on the wall.

(etc.)

Continuum-hypothesis version (by Noam Elkies, Jan. 2006):

Beth-1 bottles of beer on the wall, Beth-1 bottles of beer;
Take Aleph-1 down and pass them around:
There may be no bottles of beer on the wall.
(The End)

Complex-number version 1 (2008):

100 bottles of beer on the wall, 100 bottles of beer;
If i of those bottles should happen to fall,
100- i bottles of beer on the wall.

100- i bottles of beer on the wall, 100- i bottles of beer;
If i of those bottles should happen to fall,
100-2 i bottles of beer on the wall.
(etc.)

Complex-number version 2 (2009):

100 i bottles of beer on the wall, 100 i bottles of beer;
If i of those bottles should happen to square,
99 i -1 bottles of beer on the wall.

99 i -1 bottles of beer on the wall, 99 i -1 bottles of beer on the wall,
If i of those bottles should happen to square,
98 i -2 bottles of beer on the wall.
(etc.)

Fibonacci version (by Doug McKenna, May 2010):

$F(n)$ bottles of beer on the wall, $F(n)$ bottles of beer;
If $F(n-2)$ of those bottles should fall,
 $F(n-1)$ bottles of beer on the wall.

$F(n-1)$ bottles of beer on the wall, $F(n-1)$ bottles of beer;
If $F(n-3)$ of those bottles should fall,
 $F(n-2)$ bottles of beer on the wall.

$F(n-2)$ bottles of beer on the wall, $F(n-2)$ bottles of beer;
If $F(n-4)$ of those bottles should fall,
 $F(n-3)$ bottles of beer on the wall.
(etc.)

NB: For shorter trips, swap the indices in the second and third lines of each verse :)

Factorial version (July 2010):

$n!$ bottles of beer on the wall, $n!$ bottles of beer;
If $(n-1)/n$ of those bottles should happen to fall,
 $(n-1)!$ bottles of beer on the wall.

$(n-1)!$ bottles of beer on the wall, $(n-1)!$ bottles of beer;
If $(n-2)/(n-1)$ of those bottles should happen to fall,
 $(n-2)!$ bottles of beer on the wall.
(etc.)

Topological version (August 2010):

One full Klein bottle of beer on the wall, one full Klein bottle of beer;
If that bottle should happen to spill,
Half of a bottle of beer to refill.

A half-full Klein bottle of beer on the wall, a half-full Klein bottle of beer;
If that bottle should happen to spill,
Half of a bottle of beer to refill.

A half-full Klein bottle of beer on the wall, a half-full Klein bottle of beer;
If that bottle should happen to spill,
Half of a bottle of beer to refill.
(etc.)

Statistical version (by Larry Lesser & DAB, September 2010):

100 bottles of beer on the wall
plus or minus 1 bottle;
If one of those bottles should happen to fall,
Can't reject H_0 of 100 at all.

Non-mathematical Variations

Early non-mathematical variation (by Scott Kim?):

A hundred choruses in the song, a hundred in the song;
If one of those choruses happens to end.
(The End)

Non-mathematical variations (by Scott Kim, October 1997):

A hundred bottles of beer on the wall, a hundred bottles of beer;
If the last letter should happen to fall,
A hundred bottles of beer on the wall.
(etc.)

Four lines of lyrics on the wall, four lines of lyrics;
If one of the lines should happen to fall,
(The End)

A hundred bottles of beer on the wall, a hundred bottles of beer;
If another wall should happen to be built,
A hundred bottles of beer on two walls.
(etc.)

Instrumental version (March 2006):

[no lyrics]

Russell's paradox version (work in progress):

A hundred bottles of beer on the wall, a hundred bottles of beer;
If math logic's foundations should happen to fall,
?? [suggestions for completing this version will be appreciated!]

Acknowledgments

Thanks to Richard Byrd, Matthew Dovey, Noam Elkies, Scott Kim, Larry Lesser, and Doug McKenna for contributing both versions of the song and enthusiasm about the idea. Thanks also to the editors of *Math Horizons* for several improvements.

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